

AN INVERSE KINEMATIC PROBLEM WITH INTERNAL SOURCES

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ABSTRACT. Given a bounded domain M in \mathbb{R}^n with a conformally Euclidean metric $g = \rho dx^2$, in this paper we consider the inverse problem of recovering a semigeodesic neighborhood of a domain $\Gamma \subset \partial M$ and the conformal factor ρ in the neighborhood from the travel time data (defined below) and the Cartesian coordinates of Γ . We develop an explicit reconstruction procedure for this problem. The key ingredient is the relation between the reconstruction and a Cauchy problem of the conformal Killing equation.

1. INTRODUCTION

Let (M, g) be a bounded domain in \mathbb{R}^n , $n \geq 2$ with smooth boundary ∂M . We assume g is conformal to the Euclidean metric, i.e. $g = \rho dx^2$, where ρ is a positive smooth function on M and $dx^2 = (dx^1)^2 + \dots + (dx^n)^2$ is the Euclidean metric. Let Γ be a domain in ∂M (in particular Γ could be ∂M), from each $x' \in \Gamma$, there is a unique geodesic $\gamma_{x'}(t)$ with $\gamma_{x'}(0) = x'$, $\dot{\gamma}_{x'}(0) = \nu(x')$, where $\nu(x')$ is the inward unit normal vector to ∂M at x' w.r.t. the metric g . Moreover, since γ is a geodesic of unit speed w.r.t. the metric g , we have $\rho(\gamma)|\dot{\gamma}|^2 = 1$, where $|\cdot|$ is the Euclidean norm.

There is a positive smooth function $T(x')$ on Γ such that for each $x' \in \Gamma$, the geodesic $\gamma_{x'}$, which is orthogonal to ∂M at x' , is defined on the interval $[0, T(x')]$. Let

$$D := \{(x', t) : x' \in \Gamma, 0 \leq t < T(x')\},$$

we consider the map $\gamma : D \rightarrow \gamma(D)$, $\gamma(x', t) := \gamma_{x'}(t) = x$. Generally such a map is not a diffeomorphism, for example given the Euclidean disk with radius r , let Γ be a domain of the boundary, then γ is not a diffeomorphism if $T(x') > r$. Thus we modify $T(x')$, i.e. D , so that $\gamma : D \rightarrow \gamma(D)$ is a diffeomorphism. Under this assumption, D actually provides a semigeodesic coordinate system (or boundary normal coordinates) for $\gamma(D)$.

Now given a point $x \in \gamma(D)$, if $x' \in \Gamma$ such that $x = \gamma(x', t)$ for some $t \in [0, T(x')]$, let $U(x') \subset \Gamma$ be a neighborhood of x' . Moreover, we fix $U(x')$ for all $x \in \gamma_{x'}([0, T(x')])$, and $U(x')$ can be arbitrarily small. Notice that the choices of $U(x')$ somehow depend on a priori knowledge of x , i.e. we need to know the geodesic projection $x'(x) \in \Gamma$ of any $x \in \gamma(D)$. We define the *travel time data* w.r.t. D by

$$\Omega(D) := \{(\tau(x, x''), x'(x)) : x \in \gamma(D), x'' \in U(x')\},$$

where $\tau(x, x'') := \text{dist}_g(x, x'')$. In this paper, we consider the problem of recovering the neighborhood $\gamma(D)$ and the conformal factor ρ in $\gamma(D)$ from the travel time data $\Omega(D)$. To solve the problem, we need some extra inverse data, namely we assume the Cartesian coordinates of Γ is known. This is a reasonable assumption, since any rigid transformation of the domain M does not change the travel time data. We call the problem the *inverse kinematic problem with internal sources*. It's worth pointing out that we do not put any assumption on the convexity of the boundary (or Γ). Uniqueness for this inverse problem was proved by Yu. E. Anikonov [1]. In this paper we give a reconstruction procedure, based on conformal Killing vector fields.

Theorem 1.1. *Let M be a bounded domain in \mathbb{R}^n , $n \geq 2$ and $g = \rho dx^2$ be a conformally Euclidean metric on M . Let Γ be a domain in ∂M , then there exists a semigeodesic coordinate system D such that $\gamma : D \rightarrow \gamma(D) \subset \mathbb{R}^n$ is a diffeomorphism and $\Gamma = \gamma(\{t = 0\})$. We develop a reconstruction procedure of the diffeomorphism γ and the conformal factor ρ in $\gamma(D)$ from the travel time data $\Omega(D)$ and the Cartesian coordinates of Γ .*

Generally one can not expect to reconstruct ρ on the whole manifold, as the necessary assumption that γ is a diffeomorphism. However, if $M \setminus \gamma(D)$ has empty interior, we reconstruct the domain M and the conformal factor ρ globally by taking limit. The example mentioned above satisfies the assumption, and it is easy to see that $M \setminus \gamma(D)$ is the center of the disk if $\Gamma = \partial M$, $T(x') \equiv r$.

Corollary 1.2. *Let M be a bounded domain in \mathbb{R}^n , $n \geq 2$ and $g = \rho dx^2$ be a conformally Euclidean metric on M . Assume that there exists a semigeodesic coordinate system $\mathcal{D} = \{(x', t) : x' \in \partial M, t \in [0, T(x')]\}$ such that $\gamma : \mathcal{D} \rightarrow \gamma(\mathcal{D}) \subset \mathbb{R}^n$ is a diffeomorphism with $\gamma(\{t = 0\}) = \partial M$, and $M \setminus \gamma(\mathcal{D})$ has empty interior. Then there is a reconstruction procedure of the domain M and the conformal factor ρ from the travel time data $\Omega(\mathcal{D})$ and the Cartesian coordinates of ∂M .*

The inverse kinematic problem arose in geophysics in an attempt to determine the substructure of the Earth by measuring at the surface of the Earth the travel times of seismic waves. In application the conformal factor ρ corresponds to $1/c^2(x)$, where $c(x)$ is the sound speed (index of refraction). It goes back to [7, 19] who considered the case of a radial metric conformal to the Euclidean metric. The case considered above corresponds to an isotropic media, however it has been realized later that the inner core of the Earth might exhibit anisotropic behavior [4]. The geometric version of the problem is related to the boundary rigidity problem and its linearization, namely the geodesic ray transform, see [16] for a recent survey.

However, in current paper we also take use of the internal data (data from internal sources), so that we can reconstruct the geometry explicitly. In [9, 8] an isometric copy of a compact Riemannian manifold was recovered from the set of boundary distance functions $\{r_x(y) := \text{dist}(x, y) : x \in M, y \in \partial M\}$, which is also internal data. Such internal data is also related to the broken geodesic flow which consists of two geodesic segments sharing a common end point inside the manifold, see e.g. [8, 10]. A related reconstruction problem with different assumptions was considered

in [6] by reducing the travel time data to measurements of the shape operators of the wave fronts of waves diffracted from interior points. Different from our method, their approach treated the case of two dimensions and the case of three and higher dimensions separately.

Notice that the statement of Theorem 1.1 actually shows that this is a local problem, i.e. for a point $x = \gamma(x', t)$, we only need the travel time data from x to an arbitrarily small neighborhood $U(x') \subset \Gamma$. If the function $T(x')$ is also uniformly small, the problem can be formulated just near one boundary point. Similar to the local version of the problem, the local boundary rigidity problem and local geodesic ray transform were considered in [18, 17], and a generalization to local ray transforms along arbitrary smooth curves was studied in [20].

As mentioned above, our arguments give a reconstruction procedure for the diffeomorphism γ and the conformal factor ρ . The reconstruction procedure consists of two steps: step 1 (Section 2) devotes to the recovery of a semigeodesic (isometric) copy of the metric g and the boundary restriction of the conformal factor from the inverse data; in step 2 (Section 3), we reconstruct γ and ρ by studying the relation between γ and conformal Killing vectors on the semigeodesic copy D .

Notice also that the inverse dynamical problem for the wave equations (with boundary data) may be reduced by the boundary control method to the inverse kinematic problem with internal sources and then to the Yamabe problem [3]. It gives in our case the Cauchy problem for the Laplace operator. We use another approach (using conformal Killing vector fields) that gives stability for dimensions $n > 2$ [13].

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2. RECOVERY OF THE SEMIGEODESIC COPY OF THE METRIC

Given $x \in \gamma(D)$, there is a unique geodesic $\gamma_{x'}$ (normal to ∂M) and $0 \leq t < T(x')$ such that $x = \gamma(x', t)$. Here $x'(x)$ is the geodesic projection of x on Γ , so $t(x) = \tau(x, x'(x))$ is the distance from x to the boundary. Thus the travel time data $\Omega(D)$ uniquely determines the semigeodesic coordinates of points in $\gamma(D)$.

Let the pair $(x'(x), t(x))$ be the semigeodesic coordinate of the point x . Thus for any point $y = (x', t) \in D$, the function $\tau(\gamma(x', t), x'')$ is known. Define

$$\lambda((x', t), x'') := \tau(\gamma(x', t), x''),$$

then $\lambda(y, x'')$, $y = (x', t)$ is the distance between points $y \in D$ and $x'' \in U(x')$ in the metric $\tilde{g} := \gamma^*(g)$. Note that we identify Γ with $\Gamma \times \{0\}$. We call \tilde{g} the *semigeodesic copy* of the metric g .

We first recover the metric \tilde{g} . Notice that γ , now as an isometry, sends geodesics to geodesics, this implies

$$\tilde{g}_{kn}(x', 0) = \delta_{kn}, \quad x' \in \Gamma, \quad 1 \leq k \leq n,$$

where δ_{ij} is the Kronecker delta. In local coordinates, $y = (y^1, \dots, y^n)$, where $y^n = t$, one has the following eikonal equation

$$1 = |\nabla \tau(x, x'')|_g^2 = |\tilde{\nabla} \lambda(y, x'')|_{\tilde{g}}^2 = \tilde{g}^{ij}(y) \frac{\partial \lambda(y, x'')}{\partial y^i} \frac{\partial \lambda(y, x'')}{\partial y^j}, \quad x'' \in U(x'). \quad (1)$$

Note that if y' is close enough to y , and $x'' \in U(x'(y))$ sufficiently close to $x'(y)$, then $x'' \in U(x'(y'))$ too. (1) gives a family of linear algebraic equations w.r.t. the contravariant components $\tilde{g}^{ij}(y)$ of the metric \tilde{g} . For $y \in D$ with $y^n > 0$, it is known that $\tilde{g}^{ij}(y)$ can be recovered by the knowledge of $\tilde{g}^{ij}(y) \partial_{y^i} \lambda(y, x'_k) \partial_{y^j} \lambda(y, x'_k)$ for $N = n(n+1)/2$ “generic” points $x'_k \in U(x'(y))$, $k = 1, 2, \dots, N$, see e.g. [14, 15]. Such N generic points always exist in a neighborhood of $x'(y)$ in Γ . Thus $\tilde{g}^{ij}(y)$ is determined for $y \in D$, $y^n > 0$. By the smoothness of \tilde{g} , it also recovers $\tilde{g}^{\alpha\beta}(x', 0)$, $1 \leq \alpha, \beta \leq n-1$, $x' \in \Gamma$. So the metric \tilde{g} is uniquely determined.

Notice that the boundary restriction of the isometry γ is given, i.e. $x' = x'(y^1, \dots, y^{n-1}, 0) = (x'^1, \dots, x'^n)$ as a point in \mathbb{R}^n is known, we recover the conformal factor ρ on Γ . Since $\tilde{g} = \gamma^*(g)$ is the pullback, we have

$$\tilde{g}_{\alpha\beta}(x', 0) = \frac{\partial \gamma^k}{\partial y^\alpha} \frac{\partial \gamma^l}{\partial y^\beta} \rho(x') \delta_{kl} = \rho(x') \sum_{k=1}^n \frac{\partial x'^k}{\partial y^\alpha} \frac{\partial x'^k}{\partial y^\beta}, \quad \alpha, \beta = 1, \dots, n-1. \quad (2)$$

Thus equation (2) and the knowledge of $\tilde{g}|_{t=0}$ together determine $\rho|_\Gamma$.

3. RECOVERY OF THE ISOMETRY AND THE CONFORMAL FACTOR

To recover the conformal factor ρ , we need to solve the *pullback problem*, i.e. to find the map γ . To this end, we need some knowledge of conformal Killing vector fields. Recall that a vector field u is called a conformal Killing vector field if it satisfies the conformal Killing equation (in covariant form)

$$Ku := \sigma \nabla u - g \delta u / n = 0,$$

where $\sigma \nabla$ is the symmetric part of the covariant derivative ∇ , δ is the divergence. They are exactly those vector fields whose flows preserve the conformal structures of the manifolds. In local coordinates, the conformal Killing equation has the form (for covariant components)

$$\frac{1}{2} \left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} - 2 \Gamma_{ij}^k u_k \right) - \frac{1}{n} g_{ij} (g^{kl} \frac{\partial u_l}{\partial x^k} - g^{kl} u_m \Gamma_{kl}^m) = 0.$$

Thus the conformal Killing equation is equivalent to a system of first order partial differential equations.

However, not all of the metrics admit conformal Killing vector fields, actually for $n \geq 3$, a “generic” metric does not possess any non-trivial conformal Killing vector fields, see e.g. [2, 11]. On the other hand, note that the metric g is conformal to the Euclidean metric, thus they share the same set of conformal Killing vector fields. When $n = 2$, in the Cartesian coordinates (x^1, x^2) all conformal Killing vector fields of the Euclidean metric have the form $u = (u^1, u^2)$, where u^1 and u^2 are conjugate

harmonic functions. In the case $n > 2$, the contravariant components of u in the Cartesian coordinates (x^1, \dots, x^n) are given by

$$u^i(x) = a_0 x^i + (Ax)^i - b^i |x|^2 + 2x^i(b, x) + c^i,$$

where a_0 is a real constant, A is a $n \times n$ skew-symmetric constant matrix, b and c are vectors in \mathbb{R}^n .

Now we are in a position to recover the map γ and the conformal factor ρ . Let $e_{(j)} = \frac{\partial}{\partial x^j}$, $j = 1, \dots, n$ be the standard basis vectors in \mathbb{R}^n . It is easy to see that they are conformal Killing vector fields in Euclidean metric, thus also conformal Killing vector fields for g . Then $u_{(j)} = \gamma^* e_{(j)}$, $j = 1, \dots, n$ are conformal Killing vector fields for the metric $\tilde{g} = \gamma^* g$. This implies $u_{(j)}$, $j = 1, \dots, n$ satisfy the conformal Killing equation

$$Ku_{(j)}(y) = 0, y \in D, j = 1, \dots, n.$$

It is known that $u_{(j)}$ is uniquely determined by the Cauchy data $\{u_{(j)}(x', 0) : x' \in \Gamma\}$, see e.g. [12, 5]. Thus we calculate the Cauchy data of $u_{(j)}$ first.

Since we have already recovered the semigeodesic copy \tilde{g} , we denote the dual vector of $u_{(j)}$ by $u^{(j)}$. In the mean time, we denote the dual vector of $e_{(j)}$ under the metric $g = \rho dx^2$ by $w^{(j)}$, then $w^{(j)} = \rho dx^j$. In local coordinates, the equality $u_{(j)} = \gamma^* e_{(j)}$ means (for covariant components)

$$u_i^{(j)}(y) = w_k^{(j)} \frac{\partial x^k}{\partial y^i} = \rho \frac{\partial \gamma^j(y)}{\partial y^i}.$$

This observation is crucial in our reconstruction procedure, it relates the isometry γ to the conformal Killing vector fields on the semigeodesic copy D . Thus at $y^n = t = 0$,

$$u_\alpha^{(j)}(x', 0) = \rho(x') \frac{\partial \gamma^j}{\partial y^\alpha}(x', 0) = \rho(x') \frac{\partial x'^j}{\partial y^\alpha}, \quad \alpha = 1, \dots, n-1.$$

Since $\rho(x')$ and $\gamma(x', 0)$ are known for $x' \in \Gamma$, $u_\alpha^{(j)}(x', 0)$ are determined. To determine the value of $u_n^{(j)}$ at $t = 0$, notice that $\dot{\gamma}_{x'}(0) = \nu(x')$, so

$$u_n^{(j)}(x', 0) = \rho(x') \frac{\partial \gamma^j}{\partial t}(x', 0) = \rho(x') \nu^j(x').$$

However, ν is a unit vector w.r.t. metric $g = \rho dx^2$, if we denote the inward unit normal vector on ∂M w.r.t. the Euclidean metric by ν_0 , then $\nu = \frac{1}{\sqrt{\rho}} \nu_0$, i.e.

$$u_n^{(j)}(x', 0) = \sqrt{\rho(x')} \nu_0^j(x').$$

Fortunately, the Cartesian coordinates of the hypersurface Γ are given, thus ν_0 as the normal vector to Γ is known. Together with the knowledge of $\rho|_\Gamma$, we recover $u_n^{(j)}|_{t=0}$ too.

From the Cauchy data, we uniquely recover the conformal Killing vector fields $u_{(j)} = \gamma^* e_{(j)}$, equivalently the dual vector $u^{(j)}$. Now by defining $v = (v^1, \dots, v^n)$,

$$v^j(x', t) := u_n^{(j)}(x', t) = \rho(\gamma(x', t)) \frac{\partial \gamma^j(x', t)}{\partial t},$$

we have $v = \rho(\gamma) \dot{\gamma}$. In the mean time, notice that

$$|v|^2 = \rho^2(\gamma) |\dot{\gamma}|^2 = \rho(\gamma) \quad (\text{since } |\dot{\gamma}|_g = 1), \quad (3)$$

we obtain

$$\dot{\gamma} = \frac{v}{|v|^2}.$$

This implies

$$\gamma(x', t) = \int_0^t \dot{\gamma}_{x'}(t) dt + x' = \int_0^t \frac{v}{|v|^2}(x', t) dt + x',$$

i.e. we recover the geodesics $\gamma_{x'}(t)$, therefore the diffeomorphism $\gamma : D \rightarrow \gamma(D)$, namely the range $\gamma(D)$. Moreover, by (3)

$$\rho(\gamma(x', t)) = |v(x', t)|^2,$$

the conformal factor $\rho|_{\gamma(D)}$ is determined.

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